On the n-body problem, chaos and computability.

1. Introduction

The Solar system and the Sky has fascinated the human mind for thousands of years. Ancient cultures have followed the seasons, the phases of the moon and five planets were known from antiquity. During many centuries was the Sky thought to be given once and for all and no changes were expected. Comets, like Halley´s, in year 1066 was a sign in the Sky before the battle at Hastings. The supernova explosion in July 1054 was observed at least in China, by Indian tribes in Mexico and Arizona, by a medical doctor in Cairo who thought that it was a sign of a plague, and possibly by many more. It is not surprising that Celestial Mechanics was an important part of mathematics for more than three centuries. The SETI project, which started optimistically by listening to radio signals at different frequencies, among them the "spin-flip" signal at 1420 MHz from interstellar neutral hydrogen, is now a major field of astronomy using different techniques to identify stars with planetary systems. Even the European Extremely Large Telescope (E-ELT) being projected holds distant planetary research as one of its most important topics.

The discovery of exotic planets around stars like our own Sun in the early 1990ies started a new research field in astronomy which by now has discovered over 700 verified planets around hundreds of stars [see Schneider]. Planets with masses between ~1 and ~10000 Earth masses have been discovered, so far the most massive has m sin i ≈ 31 M_J . A planet with a mass exceeding 31 M_J should according to common wisdom burn its deuterium and in that sense be a light brown dwarf star during some period of its lifecycle. Planets have also been found in binary star systems, for example Kepler16b, which may open a new page in the applications of perturbative calculations of orbits. Irregular or chaotic behavior is seen in long term evolution computations in the Solar system and have been studied using several methods, mostly symplectic calculations. Authors have found chaotic behavior and instabilities over the period of more than $o(10^8)$ yr. Resonances between neighboring planets and moons are frequent in the Solar system and have also been found in some of the distant planetary systems.

In this short commentary we will discuss some aspects of the n-body problem, the occurrence of resonances and chaotic behavior in celestial mechanics but also pose the question of computability, or the question "to what extent allow the global existence proofs [Sundman 1909, Wang 1991] a long term computation of orbits in the planetary systems".

2. The n-body problem

Johann Kepler published his three (or rather four) famous laws that: i) planets move along ellipses with the Sun at focus, ii) that the radius vector covers equal areas in equal times (=conserved angular momentum) and iii) that the period is proportional to the longer semi-axis to power $\frac{3}{2}$ [Kepler, 1609, 1619]. Isaac Newton gave six decades later a physical explanation for them using his law of universal gravitation and mathematical methods. In his "De Motu" of 1684 Newton stated that "the planets having a focus in the center of the Sun" whereas in his "Philosophia Naturalis Principia Mathematica" [Newton, 1684 and 1687] he obviously knew that it *is in the center of mass* of the Solar system. However at several places in Principia remains his old "mistake" in some propositions and theorems. The full description of the two-body problem that the trajectories can be any of the conic sections was later given by Johann Bernoulli [Bernoulli, 1710].

The three-body problem was not treated by Newton but he mentioned in § 66 of Principia that it "is beyond the power of the human mind". The three body problem in a restricted form was treated by Leonard Euler in 1767 [Euler, 1772] and Jean-Louis Lagrange in the late eighteenth century and some exact solutions were found. The case of the Trojan asteroids locked to Jupiter, and observationally discovered in 1906 and later, was discussed already by Lagrange in 1772. He also found a "proof" of the stability of the Solar system by showing that the perturbation equation for the longer semi-axis a had no secular growth [Lagrange, 1772 and 1788]. The "proof" was later of course found to be insufficient. The three-body problem was studied intensely for more than a century before an existence proof on the solution for all times appeared. Although not solving the problem given in King Oscar II Prize in 1890 Henri Poincare was awarded the prize mainly for his new methods of analyzing dynamical systems [Poincare, 1890]. In late nineteenth century Heinrich Bruns and Poincare [Bruns 1887, Poincare, 1890] proved a theorem stating that the only integrals of the three-body problem (and the n-body problem) are the ten classical integrals, six for center of mass motion, three for angular momentum and one for the energy. In 1909 came the existence proof for the solution of the three-body problem for all times when Karl Sundman showed that all coordinates can be expressed in series of time to power ⅓. Using an ingenious regularization transformation $dt = \left(1 - e^{-r_0/l}\right)\left(1 - e^{-r_1/l}\right)\left(1 - e^{-r_2/l}\right) d\omega$ he redefined time t into ω . The distances r_i denote the three mutual distances and l is a lower bound. The coordinates of the bodies and time are now analytic functions of ω within a infinitely long band of finite width 2Ω in the ω-plane, symmetrically above and below the real ω-line. Using a transformation introduced earlier by Henri Poincare, $\omega = \frac{2\Omega}{\pi} \ln \frac{1+z}{1-z}$, Sundman was able to transform the infinite band onto the interior of the unit circle $|z| = 1$ in the complex z -plane. All series expansions will then converge since $|z| < 1$, hence the solution exists for all times. The work was published in two articles in the early 20th century by Sundman [Sundman, 1907 and 1909] in the Finnish Acta Societatis Scientiarum Fennicae. It took some time for the mathematically minded astronomers and mathematicians to react but when Sundman, with some help from his mathematician friend Ernst Lindelöf, published a restructured article in Acta Mathematica in 1912 the mathematical establishment reacted and Sundman was awarded the Priz de Pontecoulant of the French Academy of Sciences in late 1913. It was however quickly obvious that the existence proof did not give anything new for numerical analysis of the three-body problem. D. Belorizky [Belorizky, 1930] estimated that $10^{8\,000\,000}$ terms are needed to match astronomical measurements and observations. Sundman was disappointed with his existence proof, already in a letter to Anders Donner in 1904 he describes a discussion with Poincare in Paris where he noted that "Poincare is satisfied with qualitative and theoretical results whereas he (=Sundman) only accepts formulas which numerically can be computed" [Källman, 2009, see also Barrow-Green, 2010]. After 1913 he never returned to the three-body problem except in lectures and early in 1912 he started the design of a mechanical computer for solving perturbative problems. We will return to this question later in terms of computability.

The n-body problem was discussed for decades without any sign of a global existence theorem when suddenly the young Chinese mathematician Qiudong Wang generalized Sundman´s theorem to the n-body problem [Wang, 1991], now without consideration of collisions. He also clarified in a later work the properties of the integral manifold of the three-body problem [Wang, 2001]. Perturbation theory methods which have been developed since the late eighteenth century and far reaching computations of the time-evolution of the Solar system have been performed. Consider a "planetary" system with one massive central body of mass M and n smaller bodies, where the i^{th} body has mass m_i. To be on the safe side let us for now assume that $\sum m_i \ll M$. The n-body Hamiltonian [see for example Wisdom and Holman, 1991 and Lecar et al, 2001]

$$H = \sum_{i=1}^{n-1} \frac{p_i^2}{2m_i} - \sum_{i<j} \frac{Gm_i m_j}{r_{ij}} \quad (1)$$

can be rewritten into a Keplerian part, the first term, and the interaction part, the two following terms, as follows

$$H = H_{Kep} + H_{int} = \sum_{i=1}^{n-1} \left(\frac{p'^2_i}{2m'_i} - \frac{Gm_i m_0}{r'_i} \right) + \sum_{i=1}^{n-1} G\, m_i m_0 \left(\frac{1}{r'_i} - \frac{1}{r_{i0}} \right) - \sum_{0<i<j} \frac{Gm_i m_j}{r_{ij}} \quad (2)$$

where the primed quantities refer to Jacobian coordinates and masses [Jacobi, 1843]. The interaction term can be approximated and developed in terms of the mass m_j of the perturbing planet

$$R_i = \sum_{j \neq i} G\, m_j \left(\frac{1}{|\mathbf{r}_j - \mathbf{r}_i|} - \frac{\mathbf{r}_j \bullet \mathbf{r}_i}{|\mathbf{r}_j - \mathbf{r}_i|^3} \right) , \quad (3)$$

where cubic and higher orders are neglected here.

Perturbation theory has worked rather well in the Solar system, now the new challenges from planetary systems with binary stars and super Jupiter planets may come as an "acid test" of the methods used. Observations of distant planetary systems with powerful equipment (VLT, ALMA, JWST, TPF and E-ELT) may yet probe these systems to surprising accuracy and detail.

3. Resonances and chaos in the Solar and distant planetary systems

It is not known when the question of stability of the solar system was first asked but it is plausible to assume that it was neither an astronomer nor a physicist who asking about it, it is more plausible to think that back then it was a philosopher or a "spiritual" person who did it. However, in the sixteenth or seventeenth century the question was most probably asked both by astronomers and physicists and for certain reasons. Over 760 planets have been found by early 2012 in hundreds of planetary system [Schneider] around distant stars of spectral class roughly from F to M, maybe even brown dwarfs of types L and T. It is therefore of uttermost interest to try to understand the long term evolution of planetary systems of different age. To get an idea about their stability or lack of it, model computations should be performed also in situations where the central star may have a mass $< M_{sun}$ and the perturbing planet has a mass $\gg M_J$, situations which may differ greatly from the case in the Solar system.

Let us now mention some of the resonances in the Solar system:

Jupiter and Saturn	2:5
Neptune and Pluto	2:3
Io, Europe and Ganymede	1:2:4
Titan and Hyperion	3:2
Dione and Enceladus	1:2
Thetys and Mimas	1:2

Simple resonances with only two bodies involved can be handled, at least if the perturbation is not too strong. The KAM theorem [see Siegel & Moser, 1971] gives the following promise: "a non-integrable system under Newton Universal Gravitation can be approximated by an integrable one if the perturbation is not too strong and if the resonances are non-overlapping". Small perturbations are believed to lead to small changes in most cases. In the cases where several bodies are involved and/or where the perturbing force is too strong, the KAM theorem gives no rescue, the *trajectories will not flow smoothly* along a multi dimensional torus. The Kirkwood gaps [see Kirkwood, 1867] are probably caused by overlapping resonances, a case where the KAM theorem does not hold. There are chaotic orbits in the Solar system, the resonance 2:3 makes the orbit of Pluto and the rotation of Hyperion chaotic. Some long term calculations indicate that both Mercury and Venus are at collision risk, mainly because there may occur rapid growth of the eccentricity leading to collisions on long term evolution of orbits [see Lecar et al, 2001 and Batygin et al, 2008 for detailed computations and discussions]. In addition to multiple or overlapping resonances we also have forced secular resonances to consider.

About twenty of the discovered distant planetary systems have planets with mass in excess of $20\ M_J$ meaning that a series expansion of the interaction term of equation 3 in powers of the perturbing mass will converge slowly, if at all. Cases where several bodies are involved in the multiple resonances are: including Jupiter, Saturn, Uranus and Neptune where $3\omega_J$ - $5\omega_S$ - $7\omega_U \approx 0$ and $3\omega_S - 5\omega_U - 7\ \omega_N \approx 0$. Resonances exist also in distant planetary systems, the planetary system around the star HD10180 has planets d and e in a 3:1 resonance and planets e and f close to a 5:2 resonance. Both HD10180 and Gliese 581 systems have similarities with the Solar system, in a sense they have a "Titius - Bode" type of distance relations between planets, maybe an effect of dynamic evolution of planetary systems. The planetary system KOI-730 has multiple resonances involving four planets in the proportions 6:4:4:3. The binary system Kepler 16 is also a challenge with its stellar masses of $0.69\ M_{sun}$ and $0.20\ M_{sun}$. It is also interesting to note that the ratio of the semi-axis of the binary, a_b, and of the planet, a_p , in Kepler 16b is ~ 0.3, the same as in the three cases studied by Kovacs [Kovacs, 2005].

4. Computability

Disappointed with the possibilities to use the existence proof for numerical computations Sundman started in 1913 to design a mechanical analog computer for perturbation computations of planetary orbits. He published a detailed design in a Festschrift for Anders Donner on his 60th birthday [Sundman, 1916]. The "perturbographe" as it was called was never built, the costs would probably have been far too large. The design is well studied on a mathematical level but there is no guarantee that this mechanical computer would have worked on the needed level of precision. The construction has tens of control sticks, cranks, tooth wheels and other mechanical parts and it is believable that large errors would build up in such a machine as well as in numerical calculations. Sundman noted that the construction would need a certain "degree of precision", but he did not perform any detailed studies on that matter. Sundman´s analog computer was designed to solve second order differential equations of the type entering in his work on small bodies acted upon by Jupiter at a 2:1 resonance [Sundman, 1902]. He estimated, maybe optimistically, that *one orbit of about twelve years of the motion of Jupiter could be achieved in seven minutes* of work on the perturbographe. Belorizky estimated that $10^{8000000}$ terms of Sundman´s series were needed to achieve the accuracy of observations, an estimate which indicates that time evolution of orbits along the thread of global existence theorems are not computable. We have to conclude that the general "mathematical" three-body or n-body problem using the series expansions given by the existence theorems are maybe not computable in finite time. The problem is related to the halting problem [Svozil, 2004], if the above is true then the "computer" will not halt. One problem is of course collisions, in Sundman´s case for the three body problem it is proved that the collisions can be handled by analytic continuation. This is not the case in the existence theorem for the n-body problem, there is no regularization at hand for collisions [Siegel, 1941] and they are not treated. A calculation along the series expansions of Sundman´ s theorem must, since it is numerical, be a recursive calculation, where one aims at a "limit", a limit which maybe is not computable.

5. Discussion

What kind of solutions would a long term computation of order $10^9\ yr$ of the Solar system evolution backward in time yield and what kind of simulations would a long term simulation of a distant planetary system forward in time produce? What kind of collisions and other drastic effects would come out of these and would the computations ever meet? Some computations indicate that the orbits of the outer planets in the Solar system are reasonably stable and that gaps between them are unstable. How many planets have left the Solar system violently and how many have entered? These questions are difficult to answer but it is possible that more computations and/or simulations of planetary systems around single or binary stars together with future Earth based or space based observations may shed some light on them. We will return later to the problem discussed in §4 in a more technical paper.

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